EXPLORATIONS OF SELF-SELECTIVE SOCIAL CHOICE FUNCTIONS

A THESIS PRESENTED BY BÜLENT ÜNEL

TO

THE INSTITUTE OF

ECONOMICS AND SOCIAL SCIENCES

IN PARTIAL FULFILLMENT OF THE

REQUIREMENTS

FOR THE DEGREE OF MASTER OF ARTS IN

ECONOMICS BILKENT UNIVERSITY

June 1999

EXPLORATIONS OF SELF-SELECTIVE SOCIAL CHOICE FUNCTIONS

A THESIS PRESENTED BY BÜLENT ÜNEL

TO

THE INSTITUTE OF

ECONOMICS AND SOCIAL SCIENCES

IN PARTIAL FULFILLMENT OF THE

REQUIREMENTS

FOR THE DEGREE OF MASTER OF ARTS IN

ECONOMICS BILKENT UNIVERSITY

June, 1999

Bülent Ünel

I certify that I have read this thesis and in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Arts in Economics.

Prof. Semih Koray
(Supervisor)

I certify that I have read this thesis and in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Arts in Economics.

Prof. Murat R. Sertel
Examining Committee Member

I certify that I have read this thesis and in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Arts in Economics.

Asst. Prof. Tarık Kara
Examining Committee Member

Approval of the Institute of Economics and Social Sciences

Prof. Ali Karaosmanoğlu
Director

To Kıymet and Bekir

ABSTRACT

EXPLORATIONS OF SELF-SELECTIVE SOCIAL CHOICE FUNCTIONS

Bülent Ünel

Department of Economics

Supervisor: Prof. Semih Koray

June 1999

In this study, we analyze self-selective social choice functions focusing on whether one can escape dictatoriality. Two ways are examined: In the first attempt, the set of social choice functions is restricted to tops only. With this restriction, self-selectivity turns out to be equivalent to dictatoriality. In the second, the set of prefence profiles restricted to single-peaked ones. Here we show that there are some self-selective social choice functions which are not dictatorial.

Keywords: Self-selectivity, tops only functions, dictatorship, single-peaked, independence of irrelevant alternatives.

ÖZET

KENDİ KENDİNİ SEÇEN SOSYAL SEÇİM KURALLARININ ÜZERİNE İNCELEMELER

Bülent Ünel

İktisat Bölümü

Tez Yöneticisi: Prof. Semih Koray

Haziran 1999

Bu çalışmada kendi kendini seçen sosyal seçim kurrallarım inceledik. İncelemenin vurgusu diktatörlük sonucundan kurtulup kurtulamıyacağı idi. İki durum incelendi: Birincisinde, sosyal seçim kurallarinin kümesi, sadece en tepedeki seçenekleri gözönünde bulunduran secim kuralları kümesine kısıtlandı. Bu kısıtlama altında da, kendi kendini seçerlilik ile diktatörlüğün eşdeğer olduğu sonucu çıktı. İkincisinde, kişilerin tercih profillerinin kümesi, tek tepeli tercih profilleri kümesine kısıtlandı. Bu durumda, diktatör olmayan ama kendi kendini seçen sosyal seçim kurallarının olduğu gösterildi.

Anahtar Kelimeler: Kendi kendini seçerlik, en tepedeki seçenekleri gözönünde bulunduran secim kuralları, diktatörlük, tek tepelilik, ilgisiz seçeneklerden bağımsızlık.

ACKNOWLEDGMENTS

I am grateful to Professors Semih Koray and Tarık Kara for their invaluable comments and discussions throughout my study. In particular I would like to thank to Prof. Kara for his helps in Latex.

Contents

Abstract iii

Özet iv

Acknowledgments v

Contents vi

1 Introduction **1**

1.1 Introduction 1

2 Characterization of τ-Self-Selective Social Choice Functions on Tops Only Domain **5**

2.1 Basic Notations and Definitons 5

2.2 Results 10

3 Domain Restriction in Preference Profiles:Single-Peaked Preferences **21**

3.1 Basic Notations and Definitions 21

3.2 Result 22

4 Conclusion **26**

4.1 Conclusion 26

Chapter 1

Introduction

1.1 Introduction

In a group of individuals, individual selfish-interest creates difficulties in aggregation of individuals' rational preference orderings over a fixed set of alternatives into a socially rational preference ordering. The usual way to deal with this problem is to design a rule which assigns a social preference ordering to each possible profile of individual preference orderings. From democratic point of view, in order to assign a meaningful social preference ordering, the rule should satisfy certain conditions. Firstly, as specified above, the rule should be defined for every profile of individual orderings. Secondly, if an alternative, say x, rises or does not fall in the ordering of each individual without any other change in those orderings and if x was preferred to another alternative y before the change in individual orderings, then x is still preferred to y. Thirdly, the rule should not prevent individuals from expressing a preference for some given alternative over another. In other words, the rule should not be imposed. Fourthly, if the relative positions of two particular alternatives in the set of individual orderings are the same, then their relative positions in the social preference ordering should also be the same. Finally, the rule should not be dictatorial (Arrow, 1950). However, Arrow showed that there does not exist any rule satisfying all of these conditions, if we have at least three alternatives.

The ultimate goal in aggregating of individual preferences into a social preference ordering (which is complete and transitive) can be regarded as determining the best alternatives for the group. However, we know that completeness and acyclicity are also enough to guarantee the existence of best alternatives. Hence, if we design a rule which directly assigns a single best alternative for each preference profile, we might escape from the impossibility result. Such a direct rule might also be more realistic than a rule which generates an entire preference ordering. In most cases the practical question of social choice is about the alternative(s) which are top ranked, rather than about the entire ranking of all the alternatives. Now what kind of conditions can be imposed on this rule, in order to make it plausible? Firstly, again the rule should be defined for every profile of individual orderings. Secondly, the rule (function) should be nondegenerate. In other words, for any alternative there should be some preference profile under which the function will choose that alternative. The final condition which can be traced back to Farquharson's work (1969), is quite interesting and realistic. He argued that Arrow assumes that individuals do not use their skills to behave strategically, they would, of course, manipulate their preferences if they can gain from doing so. With this objection, the function should be required to be nonmanipulable as well. Under these conditions, Gibbard(1973) and Satterthwaite(1975) characterized such social choice functions coming up with a rather disappointing result. In particular, they showed that any social choice function (SCF) satisfying the above conditions must be dictatorial.

Later Müller and Satterthwaite(1977) studied on the characterization of social choice function. They did not considered the manipulation of preference orderings. In this case, they imposed the following conditions which are similar to Arrow's. Firstly, again the rule(function) should be defined for every individual orderings. Secondly, for two alternatives x and y, if every individual prefers alternative x to y, then the function should not the alternative y. Finally, the rule should choose the same alternative under a new preference profile in which the relative positions of that alternative with respect to other alternatives remain the same or improved for every individual. They showed that these conditions implied dictatoriality. In other words, the combination of these three conditions with nondictatoriality condition yields an impossibility.

Having these impossibility results, again consider a group of individuals who will make a collective decision over a set of alternatives. Here another problem arises: which kind of SCFs should be employed by the group for the collective decision? One way of dealing with this problem is to seek some kind of consistency between the rule analyzed in making the collective decision and the rule utilized in choosing this rule itself. Roughly speaking, if a SCF being used by the group does not choose itself among several available SCFs in making the latter choice, then this situation reflects a certain inconsistency for that SCF. The concept of this kind of consistency first dealt with by Binmore(1975), where he considers an example showing that for a three-element alternative set inconsistencies are bound to arise at certain preference profiles. Koray(1999) introduces a general framework whichallows to deal with this notion of consistency, called self-selectivity, in a precise manner. He shows that a neutral unanimious social choice function is universally self-selective if and only if it is dictatorial. Koray and Slinko (1999) study Paretian self-selectivity where the "rival" social choice functions from among which a self-selective SCF should choose itself confined to Paretian ones. They show that if social choice function F is neutral, unanimous and Pareto-self-selective, then there is a dictator or a Paretian antidictator[1]. Koray and Slinko (1999) also extend this result from Paretian self-selectivity to π-self-selectivity where π is any social choice rule whose choice set includes the tops elements under any profile.

In this work we investigate possible ways to escape from this negative result. In the next section we present the characterization of universally self-selective social choice function on the tops only domain. That is we restrict the set of available neutral functions to the tops only functions[2]. The main result of this section is that even under this restriction we can not escape from dictatoriality. In proving dictatoriality we will present three different proofs. In the first proof, we first show that the axiom of independence of irrelevant alternatives is satisfied, hence by Koray's result it is dictatorial. In the second proof, we show that, self-selective social choice functions are monotonic, so by Müller and Satterthwaite Theorem dictatoriality follows. In the third proof, we directly show the dictatoriality of self-selective social choice functions. In the second section, we restrict the domain of preference profiles to single-peaked ones and find a family of universally self-selective SCFs which are not dictatorial.

[1]Formally, a voter $k \in N$ is a Paretian antidictator, if for every profile $\mathcal{R}$ and Pareto optimal alternatives a, b it is true that $a\mathcal{R}^k b \Rightarrow b \succ_k a$ and, hence $F(\mathcal{R})$ is the minimum of $\mathcal{R}^k$ on the set of Pareto optimal alternatives (Koray and Slinko (1999)).

[2]A function is said to be tops only, if it selects the same outcome for two preference profiles provided that their firs rows are the same

Chapter 2

Characterization of τ-Self-Selective Social Choice Functions on Tops Only Domain

2.1 Basic Notations and Definitons

Let N be a finite nonempty society. Let $\mathbb{N}$ stand for the set of natural numbers. Set $I_m = \{1, ..., m\}$ and denote the set of all linear orders on I_m by $\mathcal{L}(I_m)$ for each $m \in \mathbb{N}$. We will call a function

$$F: \bigcup_{m \in \mathbb{N}} \mathcal{L}(I_m)^N \to \mathbb{N}$$

a social choice function (SCF) if and only if, for each $m \in \mathbb{N}$ and each $\mathcal{R} \in \mathcal{L}(I_m)^N$, one has $F(\mathcal{R}) \in I_m$. The set of all social choice functions will be denoted by $\mathcal{F}$. For each $m \in \mathbb{N}$, $\mathcal{R} \in \mathcal{L}(I_m)^N$ and every permutation σ_m on I_m, we define the permuted linear order profile $\mathcal{R}_{\sigma_m}$ on I_m as follows: For all $i \in N, k, l \in I_m, k\mathcal{R}^i_{\sigma_m} l$

if and only if $\sigma_m(k)\mathcal{R}^i\sigma_m(l)$. Now $F \in \mathcal{F}$ will be called *neutral* if, for each $m \in \mathbb{N}$ and every permutation σ_m on I_m, one has

$$\sigma_m(F(\mathcal{R}_{\sigma_m})) = F(\mathcal{R}).$$

We will denote the set of all neutral SCFs by $\mathcal{N}$.

Neutrality of an *SCF* F will allow us to extend the domain of F to linear order profiles on any finite nonempty set. To do this, take any finite set A with $|A| = m \in \mathbb{N}$, where $|A|$ stands for the cardinality of A. Let $\mu : I_m \to A$ be a bijection. Any linear order profile L on A induces a linear order profile L_μ on I_m like above, where for all $k, l \in I_m$ one has $kL^i_\mu l$ if and only if $\mu(k)L^i\mu(l)$ $(i \in N)$. We simply define $F(L) = \mu(F(L_\mu))$. Notice that $F(L)$ does not depend upon what particular bijection in one uses.

Consider any $m \in \mathbb{N}, \mathcal{R} \in \mathcal{L}(I_m)^N$ and $\emptyset \neq \mathcal{A} \subset \mathcal{F}$. Now define the relations $\mathcal{R}^i_\mathcal{A}(i \in N)$ on $\mathcal{A}$ as follows: For all $F, G \in \mathcal{A}$ and $i \in N$, $F\mathcal{R}^i_\mathcal{A}G$ if and only if $F(\mathcal{R})\mathcal{R}^iG(\mathcal{R})$. We call $\mathcal{R}^N_\mathcal{A}$ *the preference profile on* $\mathcal{A}$ *induced by* $\mathcal{R}$ and simply denote it by $\mathcal{R}_\mathcal{A}$.

Given a complete preorder ρ on a finite nonempty set A, a linear order λ on A will be said to be *compatible with* ρ if and only if, for all $x, y \in A, x\lambda y$ implies $x\rho y$. Now for each $m \in \mathbb{N}, \mathcal{R} \in \mathcal{L}(I_m)^N$ and every nonempty finite subset $\mathcal{A}$ of $\mathcal{N}$, we will set $\mathcal{L}(\mathcal{A}, \mathcal{R}) = \{L \in \mathcal{L}(\mathcal{A})^N \mid L^i$ is linear order on $\mathcal{A}$ compatible with $\mathcal{R}^i_\mathcal{A}$ for each $i \in N$ $\}$, where $\mathcal{L}(\mathcal{A})$ stands for the set of all linear orders on $\mathcal{A}$, and call $\mathcal{L}(\mathcal{A}, \mathcal{R})$ *the set of all linear order profiles on* $\mathcal{A}$ *induced by* $\mathcal{R}$.

Definition 1 Given $F \in \mathcal{N}, m \in \mathbb{N}, \mathcal{R} \in \mathcal{L}(I_m)^N$ and a finite subset $\mathcal{A}$ of $\mathcal{N}$ with $F \in \mathcal{A}$, we say that F is *self-selective at* $\mathcal{R}$ *relative to* $\mathcal{A}$ if and only if there exists some $L \in \mathcal{L}(\mathcal{A}, \mathcal{N})$ such that $F = F(L)$. Moreover, we say that F is *self-selective at*

$\mathcal{R}$ if and only if F is self-selective at $\mathcal{R}$ relative to any subset $\mathcal{A}$ of $\mathcal{N}$ with $F \in \mathcal{A}$. Finally, F is said to be *universally self-selective* if and only if F is self-selective at each $\mathcal{R} \in \bigcup_{m\in\mathbb{N}} \mathcal{L}(I_m)^N$.

To clarify the concept of self-selectivity let us consider the following example.

Example 1 Let a group of individuals consist of three agents, namely a, b, c, and assume that the set of alternatives consists of three alternatives, 1, 2, and 3. Suppose that our individuals' preferences are as follows:

$\mathcal{R}$:

$\mathcal{R}^a$	$\mathcal{R}^b$	$\mathcal{R}^c$
3	1	1
2	2	3
1	3	2

Furthermore suppose that the set $\mathcal{A}$ of available SCFs is $\{F_1, F_2, F_3, F_4\}$. Let $F_1(\mathcal{R})$ select the alternative which is preferred by a majority of agents. If there is a tie, then F_1 will select the alternative that is most preferred by agent a. Furthermore, assume that for this preference profile we have $F_2(\mathcal{R}) = 2$, $F_3(\mathcal{R}) = F_4(\mathcal{R}) = 3$. The complete preorder $\mathcal{R}_{\mathcal{A}}$ on $\mathcal{A}$ induced by $\mathcal{R}$ is as follows:

$\mathcal{R}^a_{\mathcal{A}}$	$\mathcal{R}^b_{\mathcal{A}}$	$\mathcal{R}^c_{\mathcal{A}}$
F_3, F_4	F_1	F_1
F_2	F_2	F_3, F_4
F_1	F_3, F_4	F_2

Now $\mathcal{L}(\mathcal{A}, \mathcal{R})$ consists of 2^3 linear order profiles compatible with above complete preorder profile in each component. For example the following linear order profile

L_1 is a member of $\mathcal{L}(\mathcal{A}, \mathcal{R})$:

L_1^a	L_2^b	L_3^c
F_3	F_1	F_1
F_4	F_2	F_3
F_2	F_4	F_4
F_1	F_3	F_2

Note that $F_1(L_1) = F_1$. Thus we conclude that F_1 is self-selective at $\mathcal{R}$ relative to $\mathcal{A}$. In fact one can show that F_1 is self-selective at $\mathcal{R}$. But now consider the following preference profile $\tilde{\mathcal{R}}$:

$\tilde{\mathcal{R}}^a$	$\tilde{\mathcal{R}}^b$	$\tilde{\mathcal{R}}^c$
1	2	3
2	1	2
3	3	1

Let the set $\tilde{\mathcal{A}}$ of available SCFs be $\{F_1, F_2\}$. Moreover, assume that for this profile F_2 selects alternative 2. Note that in this case F_1 again selects 1, and $\mathcal{L}(\tilde{\mathcal{A}}, \tilde{\mathcal{R}})$ consists of just one element $\tilde{L}$ given through the following table:

$\tilde{L}_1^a$	$\tilde{L}_2^b$	$\tilde{L}_3^c$
F_1	F_2	F_2
F_2	F_1	F_1

Now clearly $F_1(\tilde{L}) = F_2$. Thus F_1 is not a self-selective at $\mathcal{R}$, and thus it is not a universally self-selective social choice function.

Definition 2 An SCF $F \in \mathcal{N}$ is said to be *unanimous* if and only if, for all $m \in \mathbb{N}, \mathcal{R} \in \mathcal{L}(I_m)^N$ and $a \in I_m$,

$$\left[\forall i \in N, \forall b \in I_m : a\mathcal{R}^i b\right] \Rightarrow F(\mathcal{R}) = a.$$

Definition 3 An SCF $F \in \mathcal{N}$ is called *Paretian* if and only if, for all $\mathcal{R} \in \bigcup_{m\in\mathbb{N}} \mathcal{L}(I_m)^N$, $F(\mathcal{R})$ is Pareto optimal with respect to $\mathcal{R}$.

Definition 4 An SCF $F \in \mathcal{N}$ satisfies *Independence of Irrelevant Alternatives* (IIA) if and only if, for all $m \in \mathbb{N}, \mathcal{R} \in \mathcal{L}(I_m)^N$,

$$[B \subset I_m, F(\mathcal{R}) \notin B] \Longrightarrow F(\mathcal{R}) = F(\mathcal{R}\mid_{I_m \setminus B}),$$

where $\mathcal{R}\mid_{I_m\setminus B}$ denotes the restriction of $\mathcal{R}$ to $I_m \setminus B$.

Definition 5 A SCF F is said to be *monotonic* if and only if

$$\forall m \in \mathbb{N} \forall \mathcal{R}, \tilde{\mathcal{R}} \in \mathcal{L}(I_m)^N : (\forall i \in N, \forall x \in I_m : F(\mathcal{R})\mathcal{R}^i x \Rightarrow F(\mathcal{R})\tilde{\mathcal{R}}^i x) \Longrightarrow F(\mathcal{R}) = F(\tilde{\mathcal{R}}).$$

Definition 6 A SCF F is said to be *strategy-proof* if and only if

$$\forall \mathcal{R}, \tilde{\mathcal{R}} \in \mathcal{L}(I_m)^N, \forall i \in N : F(\mathcal{R})\mathcal{R}^i F(\mathcal{R}^{N\setminus\{i\}}, \tilde{\mathcal{R}}^i).$$

Definition 7 Let $F \in \mathcal{N}$ and $\mathcal{R} \in \bigcup_{m\in\mathbb{N}} \mathcal{L}(I_m)^N$. Write $\tau_{\mathcal{R}} = \{(i, ArgmaxR^i) \mid i \in N\}$. F is said to be *tops only* if and only if $F(\mathcal{R}) = F(\tilde{\mathcal{R}})$, for any $\mathcal{R}, \tilde{\mathcal{R}} \in \mathcal{L}(I_m)^N$ with $\tau_{\mathcal{R}} = \tau_{\tilde{\mathcal{R}}}$.

We will denote the set of tops alternatives by $\tau(\mathcal{R})$ and the set of all neutral and tops only SCFs by $\mathcal{N}^\tau$. Let us modify Definition 1, for this set of functions.

Definition 8 Given $F \in \mathcal{N}^\tau, m \in \mathbb{N}, \mathcal{R} \in \mathcal{L}(I_m)^N$ and a finite subset $\mathcal{A}$ of $\mathcal{N}^\tau$ with $F \in \mathcal{A}$, F is said to be *τ-self-selective* if and only if F is self-selective at each $\mathcal{R} \in \bigcup_{m\in\mathbb{N}} \mathcal{L}(I_m)^N$ relative to any finite subset $\mathcal{A}$ of $\mathcal{N}^\tau$ with $F \in \mathcal{A}$.

Definition 9 An agent j is said to be *dictator* if and only if

$$\forall m \in \mathbb{N}, \forall \mathcal{R} \in \mathcal{L}(I_m)^N : F(\mathcal{R}) = Argmax\mathcal{R}^j$$

Moreover, F is said to be dictatorial SCF, if there exists some agent $j \in N$ satisfying above property.

Throughout the next section, we will use F for $F_m : \mathcal{L}(I_m)^N \to I_m$, where F_m is the restriction of F to $\mathcal{L}(I_m)^N$ with $m \in \mathbb{N}$ being kept fixed if not stated otherwise.

2.2 Results

Proposition 1 Let $F \in \mathcal{N}^\tau$ be a unanimous *SCF*. If F is τ-self-selective then F is *Paretian*.

Proof: Firstly note that, for any $m \in \mathbb{N}, \mathcal{R} \in \mathcal{L}(I_m)^N$ and $a \in \tau(\mathcal{R})$, there exists $G \in \mathcal{N}^\tau$ such that $G(\mathcal{R}) = a$.

Now assume that F is τ-self-selective. Take any $\mathcal{R} \in \mathcal{L}(I_m)^N$, set $F(\mathcal{R}) = a$. Suppose that there exists some $b \in I_m$ Pareto dominating a with respect to $\mathcal{R}$. Since a is Pareto dominated by b, then $a \notin \tau(\mathcal{R})$. Let $c \in \tau(\mathcal{R})$ and consider another profile $\tilde{\mathcal{R}} \in \mathcal{L}(I_m)^N$, which is obtained from $\mathcal{R}$, just by pushing a to the bottom of each individual preference ordering. Since F is tops only, then $F(\tilde{\mathcal{R}}) = F(\mathcal{R}) = a$. Take some $G \in \mathcal{N}^\tau$ such that $G(\mathcal{R}) = c$. It follows that also $G(\tilde{\mathcal{R}}) = c$. Set $\mathcal{A} = \{F, G\}$. Clearly, $\mathcal{L}(\mathcal{A}, \tilde{\mathcal{R}}) = \{\tilde{L}\}$, where $G\tilde{L}^iF$ for all $i \in N$. Now $F(\tilde{L}) = F$ since F is τ-self-selective. But unanimity of F implies that $F(\tilde{L}) = G$ as well, a contradiction. So, F is Paretian. ■

Proposition 2 Let $F \in \mathcal{N}^\tau$ be a unanimous *SCF*. If F is τ-self-selective, then $F(\mathcal{R}) \in \tau(\mathcal{R})$ for each $\mathcal{R} \in \mathcal{L}(I_m)^N$.

Proof: Assume that $F(\mathcal{R}) = a \notin \tau(\mathcal{R})$. Consider $\tilde{\mathcal{R}}$ which is obtained from $\mathcal{R}$ just by pushing a to the bottom of each individual's preference ordering. Since tops did not change, $F(\tilde{\mathcal{R}}) = a$. But obviously, a is Pareto dominated at $\tilde{\mathcal{R}}$ in contradiction with Proposition 1. Thus, $F(\mathcal{R}) \in \tau(\mathcal{R})$. ■

Proposition 3 Let $F \in \mathcal{N}^{\tau}$ be a unanimous τ-self-selective SCF, and $m_1, m_2 \in \mathbb{N}$ such that $m_1 \neq m_2$. If $\mathcal{R} \in \mathcal{L}(I_{m_1})^N, \tilde{\mathcal{R}} \in \mathcal{L}(I_{m_2})^N$ and $\text{Argmax}\mathcal{R}^i = \text{Argmax}\tilde{\mathcal{R}}^i$ for each $i \in N$, then $F(\mathcal{R}) = F(\tilde{\mathcal{R}})$.

Proof: If $\text{Argmax}\mathcal{R}^i = a$ for all $i \in N$, then from unanimity $F(\mathcal{R}) = a$. Since $\text{Argmax}\mathcal{R}^i = \text{Argmax}\tilde{\mathcal{R}}^i$ for each $i \in N$ by hypothesis, we have $F(\tilde{\mathcal{R}}) = a$. Now assume that $F(\mathcal{R}) = a = \text{Argmax}\mathcal{R}^i$ for some $i \in N$ and there exist some $j \in N, b \in I_{m_1}$ such that $\text{Argmax}\mathcal{R}^j = b \neq a$, and $F(\tilde{\mathcal{R}}) = b$. Take some $G \in \mathcal{N}^{\tau}$ such that $G(\mathcal{R}) = b$. Set $\mathcal{A}_1 = \{F, G\}$. Then $\mathcal{L}(\mathcal{A}_1, \mathcal{R}) = \{L_1\}$ for some $L_1 \in \mathcal{L}(\mathcal{A}_1)^N$. By τ-self-selectivity, $F(L_1) = F$. Now take some $H \in \mathcal{N}^{\tau}$ such that $H(\tilde{\mathcal{R}}) = a$. Set $\mathcal{A}_2 = \{F, H\}$. Then $\mathcal{L}(\mathcal{A}_2, \tilde{\mathcal{R}}) = \{L_2\}$ for some $L_2 \in \mathcal{L}(\mathcal{A}_2)^N$. Again by τ-self-selectivity, $F(L_2) = F$. Now define a bijection $\sigma : \mathcal{A}_1 \to \mathcal{A}_2$ such that $\sigma(F) = H, \sigma(G) = F$. By neutrality[1],

$$F(L_2) = \sigma(F(L_{1_\sigma})) = \sigma(F(L_2)) = \sigma(F) = H.$$

However, this contradicts with $F(L_2) = F$. So, $F(\mathcal{R}) = F(\tilde{\mathcal{R}})$. ■

Proposition 4 Let $F \in \mathcal{N}^{\tau}$ be a unanimous τ-self-selective SCF and $\mathcal{R} \in \mathcal{L}(I_m)^N$ with $F(\mathcal{R}) = a$. If $B \subset I_m$ is such that $a \notin B$, then $F(\mathcal{R}\mid_{I_m \setminus B}) \notin \tau(\mathcal{R}) \setminus \{a\}$.

Proof: Assume that $F(\mathcal{R}\mid_{I_m \setminus B}) = b \in \tau(\mathcal{R}) \setminus \{a\}$. Take some $G \in \mathcal{N}^{\tau}$ such that $G(\mathcal{R}) = b$. Set $\mathcal{A}_1 = \{F, G\}$. Then $\mathcal{L}(\mathcal{A}_1, \mathcal{R}) = \{L_1\}$ for some $L_1 \in \mathcal{L}(\mathcal{A}_1)^N$. By τ-self-selectivity, $F(L_1) = F$. Consider $\mathcal{R}\mid_{I_m \setminus B}$. Take some $H \in \mathcal{N}^{\tau}$ such that $H(\mathcal{R}) = a \in I_m$. Set $\mathcal{A}_2 = \{F, H\}$. Then $\mathcal{L}(\mathcal{A}_2, \mathcal{R}) = \{L_2\}$ for some $L_2 \in \mathcal{L}(\mathcal{A}_2)^N$.

[1]Here we extend neutrality. Let P be a finite set. Let $\sigma : P \to P$ be a bijection. F is said to be neutral if and only if $\sigma(F(I_\sigma)) = F(I)$, where I is a linear order profile on P and I_σ is the permuted linear order profile on I.

By τ-self-selectivity, $F(L_2) = F$. Now define a bijection $\sigma : \mathcal{A}_1 \to \mathcal{A}_2$ such that $\sigma(F) = H, \sigma(G) = F$. By neutrality,

$$F(L_2) = \sigma(F(L_{1_\sigma})) = \sigma(F(L_2) = \sigma(F) = H.$$

However, this contradicts with $F(L_2) = F$. Thus, if $a \notin B$, then $F(\mathcal{R} \mid_{I_m \setminus B}) \notin \tau(\mathcal{R})\setminus\{a\}$. ■

Proposition 5 Let $F \in \mathcal{N}^\tau$ be a unanimous τ-self-selective SCF $m \geqslant 3, |N| = n \geqslant 2, N = \{j_1, ..., j_n\}, k \in \mathbb{N}$ with $k < n$ and $a, c \in I_m$. Let $\mathcal{R}, \tilde{\mathcal{R}} \in \mathcal{L}(I_m)^N$ be such that $Argmax\mathcal{R}^i = a$, for all $i \in \{j_1, \ldots, j_k\}$ and $Argmax\mathcal{R}^i = c$, for all $i \in \{j_{k+1}, \ldots, j_n\}$; $Argmax\tilde{\mathcal{R}}^i = a$, for all $i \in \{j_1, \ldots, j_{k+1}\}$ and $Argmax\tilde{\mathcal{R}}^i = c$, for all $i \in \{j_{k+2}, \ldots, j_n\}$. If $F(\mathcal{R}) = a$, then $F(\tilde{\mathcal{R}}) = a$.

Proof: Let us consider the following four profiles: $\mathcal{R}', \breve{\mathcal{R}}, \bar{\mathcal{R}}, \hat{\mathcal{R}} \in \mathcal{L}(I_m)^N$, where we assume,without loss of generality that $j_i = i$ for each $i \in N$; $a \neq c$ and $b \notin \{a, c\}$:

$$\begin{array}{cccccccc} & 1 & & k & k+1 & k+2 & \ldots & n \\ & a & & a & c & c & & c \\ \mathcal{R}': & b & & b & a & b & & b \\ & c & & c & b & a & & a \end{array} \qquad \begin{array}{cccccccc} & 1 & & k & k+1 & k+2 & \ldots & n \\ & a & \ldots & a & c & c & & c \\ \breve{\mathcal{R}}: & b & \ldots & b & b & b & & b \\ & c & \ldots & c & a & a & \ldots & a \end{array}$$

$$\begin{array}{cccccccc} & 1 & & k & k+1 & k+2 & \ldots & n \\ & a & & a & b & c & & c \\ \bar{\mathcal{R}}: & b & & b & c & b & & b \\ & c & & c & a & a & & a \end{array} \qquad \begin{array}{cccccccc} & 1 & \ldots & k & k+1 & k+2 & \ldots & n \\ & a & \ldots & a & b & c & & c \\ \hat{\mathcal{R}}: & b & \ldots & b & a & b & & b \\ & c & \ldots & c & c & a & & a \end{array}$$

Since F is tops only $F(\mathcal{R}) = a$ implies $F(\mathcal{R}') = F(\breve{\mathcal{R}}) = a$. By Proposition 4, it follows that $F(\breve{\mathcal{R}} \mid_{\{a,c\}}) = a$. Now let us consider $F(\bar{\mathcal{R}})$.

Case 1:

Assume that $F(\bar{\mathcal{R}}) = b$. Then from Proposition 4, it follows that $F(\bar{\mathcal{R}} \mid_{\{a,b\}}) = b$. Note that the relative positions of a and b in $\bar{\mathcal{R}}$ are the same as those of a and c in $\breve{\mathcal{R}}$. Thus, if we combine this with the neutrality of F we get $F(\breve{\mathcal{R}} \mid_{\{a,c\}}) = c$. But this contradicts with $F(\breve{\mathcal{R}} \mid_{\{a,c\}}) = a$.

Case 2:

Assume that $F(\bar{\mathcal{R}}) = c$. Clearly, by Proposition 4 it follows that $F(\bar{\mathcal{R}} \mid_{\{a,c\}}) = F(\breve{\mathcal{R}} \mid_{\{a,c\}}) = c$. But this will again contradict with $F(\breve{\mathcal{R}} \mid_{\{a,c\}}) = a$.

Thus $F(\bar{\mathcal{R}}) = a$. Since F is tops only, we have $F(\hat{\mathcal{R}}) = a$. Again by Proposition 4, $F(\hat{\mathcal{R}} \mid_{\{a,c\}}) = a$. Joining this with Proposition 3, we get $F(\tilde{\mathcal{R}}) = a$. ■

Corollary 1 Let $F \in \mathcal{N}^{\tau}$ be a unanimous SCF and $\mathcal{R}, \tilde{\mathcal{R}} \in \mathcal{L}(I_m)^N$. Assume that $F(R) = a$ and $Argmax\tilde{\mathcal{R}}^i = Argmax\mathcal{R}^i$ for all $i \in N \backslash \{j\}$, where j is an agent with $Argmax\mathcal{R}^j \neq a$ and $Argmax\tilde{\mathcal{R}}^j = a$. If F is τ-self-selective, then $F(\tilde{\mathcal{R}}) = a$.

Proof: Assume that $Argmax\mathcal{R}^j = c \neq a$.If $Argmax\mathcal{R}^i = a$ for all $i \in N \backslash \{j\}$, then by unanimity $F(\tilde{\mathcal{R}}) = a$.

So let assume that $Argmax\mathcal{R}^k = b \notin \{a, c\}$ for some $k \in N$ and $F(\tilde{\mathcal{R}}) = b$. By hypothesis we have $F(\mathcal{R}) = a$ and hence by Proposition 4, $F(\mathcal{R} \mid_{\{a,b\}}) = a$. Now consider $\mathcal{R}'$ which is defined as follows: $\mathcal{R}'^i = \mathcal{R}^i \mid_{\{a,b\}}$ for all $i \in N \backslash \{j\}$ and $a\mathcal{R}'^j b$. By Proposition 5, we will have $F(\mathcal{R}') = a$. Now since $F(\tilde{\mathcal{R}}) = b$, again by Proposition 4, we have $F(\tilde{\mathcal{R}} \mid_{\{a,b\}}) = b$. But note that $\mathcal{R}' = \tilde{\mathcal{R}} \mid_{\{a,b\}}$, whence $F(\mathcal{R}') = b$, in contradiction with $F(\mathcal{R}') = a$. So, $F(\tilde{\mathcal{R}}) = a$. ■

Proposition 6 Let $F \in \mathcal{N}^{\tau}$ be a unanimous SCF. If F is τ-self-selective, then F satisfies IIA.

Proof: The crucial point in this proposition is to show that, for $\mathcal{R} \in \mathcal{L}(I_m)^N$, if $b \notin \tau(\mathcal{R})$ and $F(\mathcal{R}) = a$, then $F(\mathcal{R} \mid_{\{a,b\}}) = a$. Let us prove this.

Now consider any agent who prefers b to a. Note that if there is no such agent, then by unanimuity it follows that $F(\mathcal{R} \mid_{\{a,b\}}) = a$.

So let us suppose that there exists some agent j who prefers b to a. Since $b \notin \tau(\mathcal{R})$, there is some $c \notin \{a, b\}$ such that $Argmax\mathcal{R}^j = c$. Let us consider any agent $k \in N$:

Case 1:

If agent k prefers c to a and a to b according to $\mathcal{R}^k$, then change her preference ordering so as to make her prefer a to c and c to b.

Case 2:

If agent k prefers b to a and a to c according to $\mathcal{R}^k$, then change her preference ordering so as to make her prefer b to c and c to a.

Now consider all agents who prefer c to a and a to b according to $\mathcal{R}$. If we apply the operation described in Case 1 to all such agents, we get a new profile $\mathcal{R}'$. By the conjuction Corollary 1 and the tops onlyness, we will have $F(\mathcal{R}') = a$.

Now consider all agents who prefer b to a and a to c according to $\mathcal{R}'$. If we apply the operation described in Case 2 to all such agents, we get a new profile $\tilde{\mathcal{R}}$ from $\mathcal{R}'$. Since F is tops only and tops did not change, then $F(\tilde{\mathcal{R}}) = a$. By Proposition 4, we will have $F(\tilde{\mathcal{R}} \mid_{\{a,c\}}) = a$. Now consider $\mathcal{R} \mid_{\{a,b\}}$. Define a bijection $\sigma : \{a, c\} \to \{a, b\}$, where $\sigma(a) = a, \sigma(c) = b$. By neutrality of F it follows that $F(\tilde{\mathcal{R}} \mid_{\{a,c\}}) = F(\mathcal{R} \mid_{\{a,b\}}) = a$.

Now take any $B \subset I_m$ with $a = F(\mathcal{R}) \notin B$, and suppose that $F(\mathcal{R}) \neq F(\mathcal{R}') = b$, where $\mathcal{R}' = \mathcal{R} \mid_{I_m \setminus B}$. Then $\mathcal{R} \mid_{\{a,b\}} = \mathcal{R}' \mid_{\{a,b\}}$ with $\{a, b\} \subset \tau(\mathcal{R}')$. Now, however, $F(\mathcal{R}' \mid_{\{a,b\}}) \notin \tau(\mathcal{R}')\backslash\{b\}$, or equivalently, $F(\mathcal{R}' \mid_{\{a,b\}}) = b$ by Proposition 4, in contradiction with $F(\mathcal{R}\mid_{\{a,b\}}) = a$. ■

Corollary 2 Let $F \in \mathcal{N}^{\tau}$ be a unanimous SCF. F is τ-self-selective if and only if F is *dictatorial.*

Proof: The if part is obvious. For the only if part, since we showed that F satisfies IIA, then by Koray (1999) it follows that F is dictatorial. ■

But we will also prove directly that a neutral, unanimous, tops only, τ-self-selective SCF F is dictatorial. Before presenting the direct proof, however, we will give a second proof in which we will show that such an F is *monotonic*, and hence dictatorial by Müller-Satterthwaite Theorem.

For any $\mathcal{R} \in \mathcal{L}(I_m)^N$, let us denote the set of individuals whose best elements according to $\mathcal{R}$ are a by $S_a(\mathcal{R})$. That is, $S_a(\mathcal{R}) = \{i \in N \mid ArgmaxR^i = a\}$.

Proposition 7 Let $F \in \mathcal{N}^{\tau}$ be a unanimous τ-self-selective SCF, and $\mathcal{R}, \tilde{\mathcal{R}} \in \mathcal{L}(I_m)^N$. Let $F(\mathcal{R}) = a$. If $S_a(\tilde{\mathcal{R}}) = S_a(\mathcal{R})$, then $F(\tilde{\mathcal{R}}) = a$.

Proof: Firstly let us consider the set

$$V = \{i \in N \mid \quad Argmax\mathcal{R}^i \neq Argmax\tilde{\mathcal{R}}^i\}.$$

If $V \neq \emptyset$, then $F(\tilde{\mathcal{R}}) = F(\mathcal{R}) = a$ since F is tops only. Otherwise, take any $j \in V$. Assume that $Argmax\tilde{\mathcal{R}}^j = c$ and $Argmax\mathcal{R}^j = b$. Consider any $k \in N \backslash S_a(\tilde{\mathcal{R}})$:

Case 1:

If agent k prefers c to a and a to b according to $\mathcal{R}^k$, then change her preference ordering so as to make her prefer c to b and b to a.

Case 2:

If agent k prefers b to a and a to c according to $\mathcal{R}^k$, then change her preference ordering so as to make her prefer b to c and c to a.

If we apply Case 1, and Case 2 to all $k \in N \backslash S_a(R)$, we will get a new profile $\hat{\mathcal{R}}$. Since tops did not change, $F(\hat{\mathcal{R}}) = F(\mathcal{R}) = a$. Now from Proposition 4, it follows that $F(\hat{\mathcal{R}} \mid_{\{a,b\}}) = a$. Let us consider a new profile $\hat{\mathcal{R}}_{(j)}$ which is obtained from $\hat{\mathcal{R}}$ just by pushing alternative c to the top in $\hat{\mathcal{R}}^j$. Consider $\hat{\mathcal{R}} \mid_{\{a,b\}}$ and $\hat{\mathcal{R}}_{(j)} \mid_{\{a,c\}}$. Now define a bijection $\sigma : \{a, b\} \to \{a, c\}$, where $\sigma(a) = a, \sigma(b) = c$. From neutrality of F we will have $F(\hat{\mathcal{R}} \mid_{\{a,b\}}) = F(\hat{\mathcal{R}}_{(j)} \mid_{\{a,c\}}) = a$. Thus it follows that $F(\hat{\mathcal{R}}_{(j)}) = a$. By continuing in this way, one can change the top elements of each agent j in $\mathcal{R}^j$ so as to make it equal to the top element of $\tilde{\mathcal{R}}^j$ for all $j \in V$. Let us denote this final profile by $\bar{\mathcal{R}}$. Clearly, because of above process $F(\bar{\mathcal{R}}) = a$. Since $Argmax\bar{\mathcal{R}}^i = Argmax\tilde{\mathcal{R}}^i$ for all $i \in N$, it follows that $F(\tilde{\mathcal{R}}) = a$. ■

Proposition 8 Let $F \in \mathcal{N}^\tau$ be a unanimous τ-self-selective SCF and $\mathcal{R}, \tilde{\mathcal{R}} \in \mathcal{L}(I_m)^N$. If $F(\mathcal{R}) = a$ and $S_a(\mathcal{R}) \subset S_a(\tilde{\mathcal{R}})$, then $F(\tilde{\mathcal{R}}) = a$.

Proof: Consider any $k \in N \backslash S_a(\tilde{\mathcal{R}})$. Change the preference ordering $\mathcal{R}^k$ so as to equal to $\tilde{\mathcal{R}}^k$. Denote the final preference profile by $\bar{\mathcal{R}}$. By Proposition 7, we will have $F(\bar{\mathcal{R}}) = a$. Now take $\bar{\mathcal{R}}$, and push alternative a to the top for all agents $i \in S_a(\tilde{\mathcal{R}})$. Let us denote this profile by $\hat{\mathcal{R}}$. By Corollary 1, we get $F(\hat{\mathcal{R}}) = a$. Now, since $Argmax\hat{\mathcal{R}}^i = Argmax\tilde{\mathcal{R}}^i$ for all $i \in N$ and F is tops only, we get $F(\tilde{\mathcal{R}}) = a$. ■

Corollary 3 Let $F \in \mathcal{N}^\tau$ be a unanimous SCF. If F is τ-self-selective, then it is monotonic.

Proof: Let $\mathcal{R}, \tilde{\mathcal{R}} \in \mathcal{L}(I_m)^N$. Assume that $F(\mathcal{R}) = a$, and in $\{x \in I_m \mid a\tilde{\mathcal{R}}^k x\} \supset \{x \in I_m \mid a\mathcal{R}^k x\}$ for all $k \in N$. Note that in this case $S_a(\mathcal{R}) \subset S_a(\tilde{\mathcal{R}})$. By Proposition 8, it follows that $F(\tilde{\mathcal{R}}) = a$. So, F is monotonic. ■

Corollary 4 Let $F \in \mathcal{N}^\tau$ be a unanimous SCF. If F is τ-self-selective, then it is dictatorial.

Proof: Since $F \in \mathcal{N}^\tau$ is a unanimous and τ-self-selecetive SCF, then by Corollary 3 F is monotonic. Hence by Müller-Sathertwaite Theorem, F is dictatorial. Note that, in fact we just proved that F_m is dictatorial. To prove that F is dictatorial we follow Koray (1999).

Now let us consider any $k > l \geqslant 3$. Let $\mathcal{R} \in \mathcal{L}(I_k)^N$ be defined as follows: For any $t \in I_{k-1}$; $t\mathcal{R}^{i_k}(t+1)$ and $(t+1)\mathcal{R}^j t$ for all $j \in N\backslash\{i_k\}$. Then $F(\mathcal{R}\mid_{I_l}) = 1$ since F satisfies IIA. But for each $j \in N\backslash i_k$, $Argmax_{I_l}\mathcal{R}\mid_{I_l} = l \neq 1$. Thus, $i_k = i_l$.

Finally, take any $\mathcal{R} \in \mathcal{L}(I_2)^N$. Define $\tilde{\mathcal{R}} \in \mathcal{L}(I_3)^N$ as follows: for any $i \in N$ and any $x, y \in I_2$, $x\tilde{\mathcal{R}}^i y$ if and only if $x\mathcal{R}^i y$; and for any $\imath \in N$ and $x \in I_2, x\tilde{\mathcal{R}}^i 3$. Then $F(\tilde{\mathcal{R}}) \in I_2$ since F is Paretian and $F(\tilde{\mathcal{R}}) = Argmax_{I_3}\tilde{\mathcal{R}}^{i_0}$. But since F also satisfies IIA and $\tilde{\mathcal{R}}\mid_{I_2} = \mathcal{R}$, we have $F(\tilde{\mathcal{R}}) = F(\mathcal{R})$. Moreover, by construction of $\tilde{\mathcal{R}}$, $Argmax_{I_3}\tilde{\mathcal{R}}^{i_0} = Argmax_{I_2}\mathcal{R}^{i_0}$, implying that $F(\mathcal{R}) = Argmax_{I_2}\mathcal{R}^{i_0}$. So i_0 is dictator when $m = 1$, we conclude that F is dictatorial. ■

Proposition 9 Let $F \in \mathcal{N}^\tau$ be a unanimous SCF . Let $\mathcal{R}, \tilde{\mathcal{R}} \in \mathcal{L}(I_m)^N$, and $F(\mathcal{R}) = a$. Assume that $S_a(\mathcal{R})\backslash\{j\} \subset S_a(\tilde{\mathcal{R}})$ for some $j \in N$ with $\text{Argmax}\tilde{\mathcal{R}}^j = b$. If F is τ-self-selective, then $F(\tilde{\mathcal{R}}) \in \{a, b\}$.

Proof: First note that, if $a = b$, then the result follows by Proposition 8. Now $\bar{\mathcal{R}} \in \mathcal{L}(I_m)^N$ be such that $\bar{\mathcal{R}}^i = \tilde{\mathcal{R}}^i$ for all $i \in N\backslash S_a(\mathcal{R})$, and $Argmax\bar{\mathcal{R}}^i = a$ for all $i \in S_a(\mathcal{R})$. By Proposition 8, $F(\bar{\mathcal{R}}) = a$. Assume that $F(\tilde{\mathcal{R}}) = c \notin \{a, b\}$. Now rearrange the order of alternatives a and c in profile $\bar{\mathcal{R}}$ in such a way that in the final profile, call it $\hat{\mathcal{R}}$, the relative positions of a and c are the same with as in $\tilde{\mathcal{R}}$ (Of course by changing $\tilde{\mathcal{R}}^j$, if necessary, so that $a\tilde{\mathcal{R}}^j c$). Clearly, since the tops did not change, we got $F(\hat{\mathcal{R}}) = a$. On the one hand, if $S_a(\tilde{\mathcal{R}}) \neq \emptyset$, then $F(\tilde{\mathcal{R}} \mid_{\{a,c\}}) = c$ by Proposition 4, implying that $F(\hat{\mathcal{R}} \mid_{\{a,c\}}) = c$ since $\tilde{\mathcal{R}} \mid_{\{a,c\}} = \hat{\mathcal{R}} \mid_{\{a,c\}}$. On the other hand, if $S_a(\tilde{\mathcal{R}}) = \emptyset$, then $S_a(\mathcal{R}) = \{j\}$. Now, however, we again have $F(\hat{\mathcal{R}} \mid_{\{a,c\}}) = c$ by Corollary 1. But this contradicts $F(\hat{\mathcal{R}}) = a$. ∎

Lemma 1 Let $F \in \mathcal{N}^\tau$ be a unanimous and τ-self-selective SCF. Let $\mathcal{R}, \tilde{\mathcal{R}}, \bar{\mathcal{R}} \in \mathcal{L}(I_m)^N$, and $F(\mathcal{R}) = a$. Assume that there is some $j \in N$ such that $\mathcal{R}^i = \tilde{\mathcal{R}}^i = \bar{\mathcal{R}}^i$ for all $i \in N \backslash \{j\}$, and $Argmax\tilde{\mathcal{R}}^j = b$, $Argmax\bar{\mathcal{R}}^j = c$ where $c \notin \{a, b\}$. Now if $F(\tilde{\mathcal{R}}) = b$, then $F(\bar{\mathcal{R}}) = c$

Proof: Note that, since $S_a(\mathcal{R})\backslash\{j\} \subset S_a(\bar{\mathcal{R}})$ and $S_b(\tilde{\mathcal{R}})\backslash\{j\} \subset S_b(\bar{\mathcal{R}})$ together with $F(\mathcal{R}) = a$ and $F(\tilde{\mathcal{R}}) = b$, we have $F(\bar{\mathcal{R}}) \in \{a, c\}$ and $F(\bar{\mathcal{R}}) \in \{b, c\}$. Thus, $F(\bar{\mathcal{R}}) = c$. ∎

Lemma 2 Let $F \in \mathcal{N}^\tau$ be a unanimous and τ-self-selective SCF. Let $\mathcal{R}, \tilde{\mathcal{R}}, \bar{\mathcal{R}} \in \mathcal{L}(I_m)^N$, and $F(\mathcal{R}) = a$. Assume that $\mathcal{R}^i = \tilde{\mathcal{R}}^i$ for all $i \in N\backslash\{j\}$, and $Argmax\tilde{\mathcal{R}}^j = Argmax\bar{\mathcal{R}}^j = b$, for some $j \in S_a(\mathcal{R})$. Now if $F(\tilde{\mathcal{R}}) = b$, then $F(\bar{\mathcal{R}}) = b$.

Proof: Consider a new profile $\hat{\mathcal{R}}$ which is defined as follows: $Argmax\hat{\mathcal{R}}^i = Argmax\tilde{\mathcal{R}}^i$, $i \in S_a(\mathcal{R})$; $Argmax\hat{\mathcal{R}}^i = Argmax\bar{\mathcal{R}}^i$, $i \in N\backslash S_a(\mathcal{R})$.

Consider $S_b(\tilde{\mathcal{R}})$. If $Argmax\hat{\mathcal{R}}^k = Argmax\tilde{\mathcal{R}}^k$ for any $k \in S_b(\tilde{\mathcal{R}})\backslash\{j\}$, then $S_b(\tilde{\mathcal{R}}) \subset S_b(\hat{\mathcal{R}})$ which, together with $F(\tilde{\mathcal{R}}) = b$, implies that $F(\hat{\mathcal{R}}) = b$ by Proposition 8. Otherwise, take any $k \in S_b(\tilde{\mathcal{R}})\backslash\{j\}$ with $Argmax\tilde{\mathcal{R}}^k \neq Argmax\hat{\mathcal{R}}^k = c$. Without loss of generality assume that $c \neq a$. Interchange the positions of b and c in $\tilde{\mathcal{R}}^k$, and denote the profile thus obtained by $\tilde{\mathcal{R}}_{(k)}$. By Proposition 9, $F(\tilde{\mathcal{R}}_{(k)}) \in \{b, c\}$. Assume that $F(\tilde{\mathcal{R}}_{(k)}) = c$. Now interchange a and b in $\tilde{\mathcal{R}}_{(k)}$ and denote this new profile by $\tilde{\tilde{\mathcal{R}}}$. Since $F(\tilde{\mathcal{R}}_{(k)}) = c$, then by Proposition 7, $F(\tilde{\tilde{\mathcal{R}}}) = c$. But, note that $F(\mathcal{R}) = a$, and $S_a(\mathcal{R}) \subset S_a(\tilde{\tilde{\mathcal{R}}})$. Thus by Proposition 8 it follows that $F(\tilde{\tilde{\mathcal{R}}}) = a$. Thus, $F(\tilde{\mathcal{R}}_{(k)}) = b$.

If we continue this process for all $k \in S_b(\tilde{\mathcal{R}})\backslash\{j\}$ we get a new profile, denote it $\breve{\mathcal{R}}$, and $F(\breve{\mathcal{R}}) = b$. Note that $S_b(\breve{\mathcal{R}}) \subset S_b(\bar{\mathcal{R}})$. Thus by Proposition 8, we get $F(\bar{\mathcal{R}}) = b$. ■

Theorem 1 *Let $F \in \mathcal{N}^\tau$ be a unanimous SCF. F is universally self-selective if and only if F is* dictatorial.

Proof: The if part is obvious.

For the only if part, let $\mathcal{R} \in \mathcal{L}(I_m)^N$, and $F(\mathcal{R}) = a \in I_m$. Pick any $j \in S_a(\mathcal{R})$. Change it with some $b \neq a$. Let us denote this profile by $\mathcal{R}_{(j)}$.

Case 1 $F(\mathcal{R}_{(j)}) = b$.

By Lemma 1, it follows that $F(\mathcal{R}_{(j)}) = b$ for any $b \in I_m$. By Lemma 2, for any $\tilde{\mathcal{R}} \in \mathcal{L}(I_m)^N$, $F(\tilde{\mathcal{R}}) = Argmax\tilde{\mathcal{R}}^j$. Hence agent j is dictator.

Case 2 $F(\mathcal{R}_{(j)}) = a$.

Set $S_a^1 = S_a\backslash\{j\}$. Now take any agent $k \in S_a^1$. Apply the above procedure for agent $k \in S_a^1$. Continuing in this way, it easily follows that there exists a dictatorial agent, since $S_a(\mathcal{R})$ is finite. Hence F is dictatorial[2]. ■

If a unanimous $F \in \mathcal{N}^\tau$ is τ-self-selective then it is dictatorial by above theorem from which it follows that F is strategy proof and monotonic (Recall that we directly proved the monotonicty). However the converse may not be true. That is, a unanimous SCF $F \in \mathcal{N}^\tau$ which is stratergy proof (or monotonic) need not be τ-self-selective. For example, let $a, b \in N$ with $a \neq b$ and for each m which is odd, let F be dictatoriality a; while for each m even, let F be dictatorship of b. Clearly, F is unanimous and strategy-proof (monotonicity), but it is not τ-self-selective

[2]Here again we just showed that F_m is dictatorial. But dictatoriality of F follows from the second part of the proof of Corollary 4.

(Koray, 1999). This situation arises because monotonicity and strategy-proofness of F treat the "components" F_m of F separately. But conjoining these conditions with IIA we get the universally *tau*-selectiveness for F. The following corollary is taken from Koray (1999).

Corollary 5 *Let $F \in \mathcal{N}^\tau$ be unanimous*

1. *F is τ-self-selective if and only if F is monotonic and satisfies IIA.*

2. *F is τ-self-selective if and only if F is straytegy-proof and satisfies IIA.*

Proof: We will only prove the first assertion, since the proof of second is similar. The "only if" part follows from above theorem. Now assume that F is monotonic and satisfies IIA. Then F_m is monotonic for each $m \in \mathbb{N}$. But then by Müller-Satterthwaite (1977) Theorem, F_m is dictatorial for all $m \geqslant 3$. Now as in the proof of the theorem, IIA implies that the dictator must be the same for all $m \geqslant 3$. In the proof of the second assertion, Gibbard(1973)-Satterthwaite(1973) Theorem is used. ■

Chapter 3

Domain Restriction in Preference Profiles:Single-Peaked Preferences

3.1 Basic Notations and Definitions

Definition 10 A complete preorder preference relation $\mathcal{R}$ is *single-peaked* with respect to the linear order $\geq$ on I_m, if there is an alternative $a \in I_m$ with the property that

$$\text{if } a \geq c > b, \text{ then } c\mathcal{R}b, \text{ and if } b > c \geq a, \text{ then } c\mathcal{R}b.$$

Definition 11 Given a linear order $\geq$ on I_m, we denote by $L(I_m)^N_{\geq}$, the collection of all complete preorders.

Given $\mathcal{R} \in L(I_m)^N_{\geq}$ and a finite nonempty subset $\mathcal{A}$ of $\mathcal{N}$, for any $G, H \in \mathcal{R}$, we say that $G \succsim H$ if and only if $G(\mathcal{R}) \geqslant H(\mathcal{R})$. Clearly $\succsim$ is a complete preorder on $\mathcal{A}$. Let $\mathcal{L}_{\geq}(\mathcal{A}, \mathcal{R})$ stands for the set of linear order profiles induced on $\mathcal{A}$ by $\mathcal{R}$ which are single peaked with respect to some linear order on $\mathcal{A}$ compatible with $\succsim$.

Definition 12 Given $F \in \mathcal{N}, m \in \mathbb{N}, \mathcal{R} \in \mathcal{L}(I_m)^N_{\geq}$ and a finite subset $\mathcal{A}$ of $\mathcal{N}$ with $F \in \mathcal{A}$, we say that F is *self-selective at $\mathcal{R}$ relative to $\mathcal{A}$* if and only if there exists some $L \in \mathcal{L}_{\geq}(\mathcal{A}, \mathcal{R})$ such that $F = F(L)$. Moreover, we say that F is *self-selective at $\mathcal{R}$* if and only if F is self-selective at $\mathcal{R}$ relative to any subset $\mathcal{A}$ of $\mathcal{N}$ with $F \in \mathcal{A}$. Finally, F is said to be *self-selective on the single-peaked domain* if and only if F is self-selective at each $\mathcal{R} \in \bigcup_{m \in \mathbb{N}} \mathcal{L}(I_m)^N_{\geq}$.

For any $i \in N$, we denote by $p_i \in I_m$ the maximal alternative for R^i, and $P = \{p_i \mid i \in N\}$. Moreover, we rank these peaks from smallest to largest with respect to $\geq$, denoting p^k the k^{th} smallest element of P[1].

3.2 Result

Proposition 10 Let F be a unanimous SCF, and $\mathcal{R} \in L(I_m)^N_{\geq}$. If $F(\mathcal{R}) = p^k$, for some fixed $k \leqslant |N|$, then F *self-selective on the single-peaked domain.*

Proof: The crucial point is to construct the appropriate L, for which $F(L) = F$. Let $\mathcal{A}$ be a finite set of SCFs. Set $\mathcal{A}_x = \{G \in \mathcal{R} \mid G(R) = x\}$, for each $x \in I_m$. We will rank $\mathcal{A}_x$ and $\mathcal{A}_y$ as follows:

$$\mathcal{A}_x < \mathcal{A}_y \Longleftrightarrow x < y.$$

For each $\mathcal{A}_x$ break ties among functions arbitrarily and fix that final order[2]. By continuing in this way, we will get a linear order on $\mathcal{A}$. Denote this final order

[1]In determining p^k, we take the multiplicityof peaks into account. That is, if, for example, $N = 5$, and $p_1 = p_3 = p_5 = 1, p_4 = 3, p_2 = 4$, then $p^3 = 1, p^4 = 3$,...and so on.

[2]For example, if we have $\mathcal{A}_1 = \{G_1, G_2, G_3\}$ and $\mathcal{A}_2 = \{G_4, G_5\}$, then we will order Gs as G_1, G_3, G_1, G_5, G_4.

as $\geq (\mathcal{A})$. Now we will construct $L \in \mathcal{L}_{\geq(\mathcal{A})}(\mathcal{A}, \mathcal{R})$ as follows: For any agent $i \in N$, consider $\mathcal{A}_{p_i}$.

i If $F \in \mathcal{A}_{p_i}$, then fix F and let L^i be such that

$$F \geq G \geq H \Longrightarrow GL^iH$$
$$F \leq G \leq H \Longrightarrow GL^iH$$

for any $G, H \in \mathcal{A}$.

ii If $F \notin \mathcal{A}_{p_i}$, then choose and fix any $G^* \in \mathcal{A}$ and let L^i be such that

$$G^* \geq G \geq H \Longrightarrow GL^iH$$
$$G^* \leq G \leq H \Longrightarrow GL^iH$$

for any $G, H \in \mathcal{A}$.

Set $P^k_> = \{x \in I_m \mid x < p^k\}$. Since $F(\mathcal{R}) = p^k$, then there are at least k agents whose peaks are either p^k or less than p^k with respect to $\geq$ on I_m. Consider any such "agent i". Since his preference ordering is single-peaked, then there is no $x \in P^k_>$ such that $x\mathcal{R}^ip^k$. Hence in the corresponding L^i, $ArgmaxL^i \leqslant F$, with respect to $\geq$ on $L(\mathcal{A})$. Obviously, from construction of L and $\geq (\mathcal{A})$, we get $F(L) = L$. ■

Let us illustrate all these in a simple example.

Example 2 Let F be a unanimous SCF. Let $\mathcal{R} \in L(I_4)^5_{\geq}$ be as follows:

a_1	a_2	a_3	a_4	a_5
1	2	2	3	4
2	3	1	2	3
3	4	3	1	2
4	1	4	4	1

Let $F(\mathcal{R}) = p^3 = 2$ and $\mathcal{A} = \{F, G, H, J, T\}$, $H(\mathcal{R}) = 1$, $G(\mathcal{R}) = 2$, $J(\mathcal{R}) = 3$, $T(\mathcal{R}) = 4$. Now according to our construction, $\geq (\mathcal{A})$ is given by $H \quad F \quad G \quad J \quad T$; while L is shown in the table below:

a_1	a_2	a_3	a_4	a_5
H	F	F	J	T
F	G	G	G	J
G	J	H	F	G
J	T	J	H	F
T	H	T	T	H

Now by using the rule F, we get $F(L) = F$. Now let change $\mathcal{R}$ to $\tilde{\mathcal{R}}$, which is as follows:

a_1	a_2	a_3	a_4	a_5
3	2	2	3	4
2	3	3	2	3
1	4	4	1	2
4	1	1	4	1

Assume that $\mathcal{A}$ is the same as above, and that this new profile H, G, J, T select the same alternatives as in the previous case. Under $\tilde{\mathcal{R}}$ the corresponding $\geq (\mathcal{A})$ will not change, but $\tilde{L}$ will be as follows:

a_1	a_2	a_3	a_4	a_5
F	G	G	F	T
J	F	F	J	J
G	J	J	G	F
H	T	T	H	G
T	H	H	T	H

Note that here, $F(\tilde{\mathfrak{R}}) = 3 \neq 2$, and furthermore $F(\tilde{L}) = F$. Hence, here we have an example where F is not dictatorial.

Chapter 4

Conclusion

4.1 Conclusion

Koray (1998) analyzed the problem of self-selectivity and found that universal self-selectivity implies dictatoriality under unanimity and neutrality assumptions for social choice functions. In this study we explored to what extent we could escape from this dictatorial result by "localizing" the notion of self-selectivity. In the first step, we restricted the set of neutral social choice functions to the tops only domain. We proved that the result was again dictatoriality. Hence we could not escape Koray's negative result. In the second step, we restricted the set of preference profiles to single-peaked ones. We considered this case, because we knew that whenever the preferences of all agents are single-peaked with respect to the same linear order a Condorcet winner existed (Mas-Colell, et.al, 1995). Hence in this domain a nondictatorial aggregation is possible. Like this result, we showed that there were some self selective social choice functions which were not dictatorial.

There are several directions in which the present work could be extended. Firstly, note that we showed that even under the restriction of the set of social choice functions to the tops only case we could not escape from dictatoriality. Hence a natural question here is what happens if we restrict the set of rivals to other more restricted domains. Secondly, in the second section we just constructed some families social choice functions which were self-selective but not dictatorial without a full characterization. A full characterization of self-selecetivity of SCFs on the single-peaked domain waits to be done. Thirdly, as Koray noted in his work, the present work can be extended to social choice correspondences.

Bibliography

[1] Arrow,K.J.[1950]."A Difficulty in the Concept of Social Welfare", Journal of Political Economy, Vol.58, pp. 328-346.

[2] Binmore, K.G.[1975]. "An Example in Group Prefernce", Journal of Economic Theory, Vol.10, pp.377-385.

[3] Farquharson, H. [1969]. "Theory of Voting", Yale University Press, New haven.

[4] Gibbard, A. [1973]. "Manipulation of Voting Schemes: A General Result", Econometrica, Vol.41, pp.587-601.

[5] Koray, S. [1999] "Self-Selective Social Choice Functions Verify Arrow and Gibbard-Satterthwaite Theorems", forthcoming in Econometrica.

[6] Koray, S., A. Slinko [1999] "On Pareto-Consistent Social Choice Functions, mimeo.

[7] Mas-Colell, A., M.D.Whinston, J.R.Green. [1995]. "Microeconomic Theory", Oxford Press.

[8] Müller., E., M.A. Satterthwaite [1977] "The Equivalence of Strong Positive Association and Strategy-Proofness", Journal of Economic Theory, Vol.14, pp.412-418.